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By

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THE DISSIPATIVITY OF A CERTAIN NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS. I

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1. Introduction

In the article sufficient conditions are given such that each solution $\vec{x} = \vec{x}(t)$ of a nonlinear system of ordinary differential equations when $t \geq T(\vec{x})$ belongs to a certain fixed bounded domain D, that is, Levinson's D-property holds [1]. At the same time, some earlier results of the author [2, 3] and S. A. Samedova [4] are generalized.

2. The principal lemma

Lemma. Let $\vec{x} = (x_1, \ldots, x_n) \in E^n$, $\vec{f}(\vec{x}) = [f_1(\vec{x}), \ldots, f_n(\vec{x})] \in C^1(D)$, where $f_i(x)$ $(i = 1, \ldots, n)$ are real and D is a convex set of a material Euclidian space E^n. Let $\lambda(\vec{x})$ and $\Lambda(\vec{x})$ be the lowest and highest characteristic numbers of the symmetrized Jacobi matrix

$$J_s(\vec{x}) = \frac{1}{2}[\vec{f}'(\vec{x}) + \vec{f}'^*(\vec{x})] = \frac{1}{2}\left[\frac{\partial f_i}{\partial x_j} + \frac{\partial f_j}{\partial x_i}\right]$$

Then for any points $\vec{x} \in D$, $\vec{x} + \Delta\vec{x} \in D$ the inequality

$$\lambda_m(\Delta\vec{x}, \Delta\vec{x}) \leq (\Delta\vec{f}(\vec{x}), \Delta\vec{x}) \leq \Lambda_M(\Delta\vec{x}, \Delta\vec{x})^*, \tag{1}$$

is satisfied, where $\Delta\vec{f}(\vec{x})=\vec{f}(\vec{x}+\Delta\vec{x})-\vec{f}(\vec{x})$; $\lambda_m=\min\lambda(\vec{\xi})$ and $\Lambda_M=\max\Lambda(\vec{\xi})$ segment $\vec{\xi}=\vec{x}+t\Delta\vec{x}\,(0\leqslant t\leqslant 1)$, connecting the points $\vec{x}$ and $\vec{x}+\Delta\vec{x}$**.

Proof. Let the points $\vec{x}$ and $\vec{x}+\Delta\vec{x}$ be fixed. Starting from the obvious equality and applying the rule of differentiation of a complex vector function, we have

$$\Delta\vec{f}(\vec{x})=\int_0^1\frac{d}{dt}\vec{f}(\vec{x}+t\Delta\vec{x})\,dt=\int_0^1\vec{f}'(\vec{\xi})\,\Delta\vec{x}\,dt,$$

where $\vec{\xi}=\vec{x}+t\Delta\vec{x}$. Hence

$$(\Delta\vec{f}(\vec{x}),\Delta\vec{x})=\left(\int_0^1\vec{f}'(\vec{\xi})\,\Delta\vec{x}\,dt,\ \Delta\vec{x}\right)=\int_0^1(\vec{f}'(\vec{\xi})\,\Delta\vec{x},\Delta\vec{x})\,dt. \qquad (2)$$

Since

$$(\vec{f}'(\vec{\xi})\,\Delta\vec{x},\ \Delta\vec{x})=(J_s(\vec{\xi})\,\Delta\vec{x},\ \Delta\vec{x}),$$

then we obviously have

$$(\vec{f}'(\vec{\xi})\,\Delta\vec{x},\ \Delta\vec{x})\geqslant\lambda(\vec{\xi})(\Delta\vec{x},\ \Delta\vec{x})\geqslant\lambda_m(\Delta\vec{x},\ \Delta\vec{x})$$

and

$$(\vec{f}'(\vec{\xi})\,\Delta\vec{x},\ \Delta\vec{x})\leqslant\Lambda(\vec{\xi})(\Delta\vec{x},\ \Delta\vec{x})\leqslant\Lambda_M(\Delta\vec{x},\ \Delta\vec{x})\ [2].$$

Therefore, inequality (1) follows from formula (2).

3. The criterion of the fixed-sign property of a matrix

Definition. The symmetric n x n matrix $A = A(\vec{x})$, which is a function of the vector $\vec{x}\in E^m$, let us call uniformly of fixed sign (uniformly positive or uniformly negative) in a given domain D if the lower bound of the eignevalues of this matrix is positive in D, or, correspondingly, the upper bound of the eigenvalues is negative in D.

Theorem 1. The symmetric n x n matrix $A = (\vec{x})$ $(\vec{x}\in D)$ is uniformly

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* $(\vec{x},\vec{y})$ is understood to be the scalar product of the real vectors $\vec{x}=(x_1,\ldots,x_n)$ and $\vec{y}=(y_1,\ldots,y_n)$, that is, $(\vec{x},\vec{y})=\sum_{i=1}^n x_iy_i$.

** For the validity of the lemma it is sufficient to assume only that the segment $\vec{\xi}=\vec{x}+t\Delta\vec{x}\,(0\leqslant t\leqslant 1)$ is entirely contained in domain D.

positively fixed in D if:

1) the Sylvester conditions are satisfied

$$\Delta_1(\vec{x}) > 0, \ldots, \Delta_n(\vec{x}) > 0,$$

where $\Delta_i(\vec{x})$ ($i = 1, \ldots, n$) are the principal diagonal minors of the determinant det $A(\vec{x})$;

2) there exists a positive number h such that

$$\frac{\Delta_n(\vec{x})}{[\mathrm{sp}\, A(\vec{x})]^{n-1}} \geq h > 0 \quad \text{when} \quad \vec{x} \in D. \tag{3}$$

The proof of this theorem is given in an earlier work of the author [2].

Corollary. The symmetric $n \times n$ matrix $A = A(\vec{x})$ ($\vec{x} \in D$) is uniformly negatively fixed in D if:

1) $(-1)^i \Delta_i(\vec{x}) > 0, \quad i = 1, \ldots, n;$

2) $$\frac{\Delta_n(\vec{x})}{[\mathrm{Sp}\, A(x)]^n} \leq -h < 0 \quad \text{when} \quad \vec{x} \in D, \tag{4}$$

where h is a positive number.

These conditions follow immediately from Theorem 1 if it is borne in mind that the uniform negative definiteness of matrix $A(\vec{x})$ follows from the uniform positive definiteness of matrix $-A(\vec{x})$.

4. Sufficient conditions for the dissipativity of the system

Theorem 2. Let

$$\frac{d\vec{x}}{dt} = \vec{f}(\vec{x}, t), \tag{5}$$

where $\vec{f}(\vec{x}, t) \in C(E^n \times I^+)$, $I^+ = (t < t < +\infty)$ and $\vec{f}(\vec{x}, t_0) \in C^1(E^n)$ for each $t_0 \in I^+$; where t (number or symbol) is ∞.

If:

1) a constant symmetric positively definite $n \times n$ matrix $A = [a_{ij}]$

is found such that the symmetric matrix

$$\vec{J}_s(\vec{x}, t) = \frac{1}{2}\{[A\vec{f}'_{\vec{x}}(\vec{x}, t)] + A\vec{f}'_{\vec{x}}(\vec{x}, t)]^*\}$$

is uniformly negatively definite in $E^n \times I^+$, that is, its highest characteristic number $\Lambda(\vec{x}, t)$ satisfies in $E^n \times I^+$ the inequality

$$\Lambda(\vec{x}, t) \leqslant -\alpha < 0, \tag{6}$$

where α is a positive constant which is not a function of $\vec{x}$ and t;

2) $\vec{f}(\vec{0}, t)$ is bounded, that is,

$$\|\vec{f}(\vec{0}, t)\| = (\vec{f}(\vec{0}, t), \vec{f}(\vec{0}, t))^{\frac{1}{2}} \leqslant c < +\infty \quad \text{when} \quad t \in I^+; \tag{7}$$

then:

1') System (5) is dissipatitive, and there exist a closed sphere $K_R = \{\|\vec{x}\| \leqslant R\} \subset E^n$, such that each solution $\vec{x} = \vec{x}(t)\ (t_0 \in I^+, \vec{x}(t_0) \in E^n)$ of System (5) possesses the property $\vec{x}(t) \in K_R$ when $T(\vec{x}) \leqslant t < +\infty$, , and, therefore, $\vec{x}(t)$ is bounded by the interval $[t_0, +\infty)$;

2') all solutions $\vec{x}(t)$ are asymptotically stable in the large when $t \to +\infty$ *; the stability is of the exponential type.

<u>Remark.</u> If the symmetrized Jacobi matrix

$$J_s(\vec{x}, t) = \frac{1}{2}[\vec{f}'_{\vec{x}}(\vec{x}, t) + \vec{f}'^{*}_{\vec{x}}(\vec{x}, t)]$$

is uniformly negatively deifned, then, obviously, it is possible to take A = E, where E is a unit n X n matrix, and, therefore, in this case

$$\vec{J}_s(\vec{x}, t) = J_s(\vec{x}, t).$$

In the particular case when System (5) is linear

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t),$$

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* It is said that the solution $\vec{x}(t)$ $(t_0 \leq t < +\infty)$ is asymptotically stable <u>in the large</u> when $t \to +\infty$ if: 1) it is according to Lyapunov when $t \to +\infty$ and 2) for any solution $\vec{y}(t)$, determined by the initial condition $\vec{y}(t_0) \in E^n$, the relation $\lim\limits_{t \to +\infty} \|\vec{y}(t) - \vec{x}(t)\| = 0$ is valid.

where $P(t) \in C(I^+)$, $\vec{f}(t) \in C(I^+)$ and $\|\vec{f}(t)\| \leqslant c$ over I^+, then for the validity of theorem 2 it is sufficient that the matrix

$$J_s(t) = \frac{1}{2}[P(t) + P^*(t)]$$

be uniformly negatively defined over I^+.

Condition (6) was used by N. N. Krasovskiy [5], who, assuming $\vec{f}(\vec{0}, t) \equiv \vec{0}$, proved the asymptotic stability in the large of the trivial solution $\vec{x} \equiv \vec{0}$.

<u>Proof 1'</u>. Let us note first of all that the derivative $\vec{f}'_{\vec{x}}(\vec{x}, t)$ is continuous and, therefore, bound in any compact domain of the set $E^n \times I^+$; therefore, the local conditions of existence and uniqueness of the solutions are satisfied for System (5).

Let us examine the quadratic form

$$v = v(\vec{x}) \equiv (A\vec{x}, \vec{x}).$$

Since matrix A is positively defined and symmetric, then, letting

$$a = \lambda_{\min}(A) \text{ and } b = \lambda_{\max}(A)$$

be the lowest and highest characteristic numbers of matrix A, we have

$$a(\vec{x}, \vec{x}) \leqslant v(\vec{x}) \leqslant b(\vec{x}, \vec{x}). \tag{8}$$

For any solution $x = x(t)$ of System (5), taking the symmetry of matrix A into account, we obtain

$$\frac{dv}{dt} = \left(A\frac{d\vec{x}}{dt}, \vec{x}\right) + \left(A\vec{x}, \frac{d\vec{x}}{dt}\right) = 2\left(A\frac{d\vec{x}}{dt}, \vec{x}\right),$$

or

$$\frac{dv}{dt} = 2(A\vec{f}(x, t), x) = 2([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)], \vec{x}) + 2(A\vec{f}(\vec{0}, t), \vec{x}). \tag{9}$$

Since condition (6) is satisfied for the derivative $\frac{\partial}{\partial \vec{x}}[A\vec{f}(\vec{x}, t)]$, then according to the principal lemma in Section 2 we have

$$([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)], \vec{x}) \leqslant -a(\vec{x}, \vec{x}) \leqslant -\frac{a}{b}v. \tag{10}$$

In addition, using inequality (7) and the Cauchy inequality, we have

$$(\vec{Af}(\vec{0}, t), \vec{x}) \leqslant \|\vec{Af}(\vec{0}, t)\| \, \|\vec{x}\| \leqslant \|A\| \, \|\vec{f}(\vec{0}, t)\| \, \|\vec{x}\| \leqslant \rho c \sqrt{\frac{v}{a}}. \qquad (11)$$

where

$$\rho = \|A\| = \max_i \sqrt{\sum_{j=1}^{n} a_{ij}^2}.$$

Therefore, from formula (9), taking inequalities (10) and (11) into account, we have

$$\frac{dv}{dt} \leqslant -\frac{2\alpha}{b} v + 2\rho c \sqrt{\frac{v}{a}} = -\frac{\alpha}{b} v - \sqrt{v} \left(\frac{\alpha}{b} \sqrt{v} - \frac{2\rho c}{\sqrt{a}} \right).$$

Therefore, if

$$v > \frac{4\rho^2 b^2 c^2}{\alpha^2 a} = v_0 .$$

and, that means, if

$$\|\vec{x}\| > \frac{2\rho bc}{\alpha a} = R, \qquad (12)$$

then

$$\frac{dv}{dt} < -\frac{\alpha}{b} v < 0 \quad \text{when} \quad t > t_0. \qquad (13)$$

Let us consider the closed sphere $K_R\{\|\vec{x}\| \leqslant R\}$. Let $v[\vec{x}(t_0)] \leqslant v_0$. Then, using inequality (13), we have, obviously, $v[\vec{x}(t)] \leqslant v_0$ when $t > t_0$. But since the ellipsoid $v(x) = v_0$ is situated within the sphere $\|\vec{x}\| = R$, then $\vec{x}(t) \in K_R$ for $t \geq t_0$; the solution $\vec{x}(t)$ is defined when $t_0 \leq t < +\infty$.

If $v[\vec{x}(t_0)] > v_0$, , then, integrating inequality (13) from t_0 to t, when $t > t_0$ we have

$$v[\vec{x}(t)] \leqslant v[\vec{x}(t_0)] e^{-\frac{\alpha}{b}(t-t_0)} < v[\vec{x}(t_0)].$$

Therefore, solution $\vec{x}(t)$ is infinitely extendable as <u>t</u> increases and cannot constantly be located outside of the ellipsoid $v(\vec{x}) \leq v_0$, that is, at some $T > t_0$ the first time we shall have $v[\vec{x}(t)] = v_0$ and, therefore, $\vec{x}(T) \in K_R$. But then $\vec{x}(t) \in K_R$ when $T \leq t < +\infty$, where

$$T < t_0 + \frac{b}{\alpha} \ln \frac{v[\vec{x}(t_0)]}{v_0}.$$

2'. Let $\vec{x} = \vec{x}(t)$ and $\vec{y} = \vec{y}(t)$ be two solutions of System (5). Assuming

$$u = (A(\vec{x} - \vec{y}), (\vec{x} - \vec{y})),$$

we have

$$\frac{du}{dt} = 2([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{y}, t)], (\vec{x} - \vec{y})).$$

Hence, using inequality (6) and the principal lemma, we obtain

$$\frac{du}{dt} \leqslant -2\alpha \|\vec{x}(t) - \vec{y}(t)\|^2 \leqslant -\frac{2\alpha}{b} u.$$

Therefore

$$u(t) \leqslant u(t_0) e^{-\frac{2\alpha}{b}(t-t_0)} \quad \text{when} \quad t > t_0,$$

or

$$\|\vec{y}(t) - \vec{x}(t)\| \leqslant \sqrt{\frac{b}{a}} \|\vec{y}(t_0) - \vec{x}(t_0)\| e^{-\frac{\alpha}{b}(t-t_0)},$$

if $t \geq t_0$. Therefore, each solution $\vec{x}(t)$ is asymptotically stable in the large when $t \to +\infty$; the stability has an exponential nature.

Corollary. At the points of the ellipsoid

$$v(\vec{x}) = v_0,$$

where $v_0 = aR^2$, for the solutions $\vec{x}(t)$ of System (5) such that $v[\vec{x}(t_0)] = v_0$, the inequality is fulfilled

$$\frac{dv}{dt} < 0 \quad \text{when } t = t_0$$

This assertion follows immediately from formulas (12) and (13).

Theorem 2a (generalization). Let the vector function $\vec{f}(\vec{x},t)$ have the properties indicated at the beginning of the formulation of theorem 2.

If:

1a) the symmetric matrix $\bar{J}_s(\vec{x}, t)$ posseses a highest characteristic number $\Lambda(\vec{x}, t)$ such that

$$\Lambda(\vec{x}, t) < -\alpha < 0 \qquad (6a)$$

when $\|\vec{x}\| \geq R_0 > 0$ and $\underline{t} < t < +\infty$,

$$\Lambda(\vec{x}, t) \leqslant \beta < +\infty \tag{6b}$$

when $\|\vec{x}\| < R_0$ and $\underline{t} < t < +\infty$, where α and β are positive numbers;

2a) the inequality

$$\|\vec{f}(\vec{0}, t)\| \leqslant c < +\infty,$$

is fulfilled, then System (5) is dissipative.

<u>Proof</u>. Retaining the symbols of theorem 2, let

$$v(\vec{x}) = (A\vec{x}, \vec{x}),$$

and let

$$\vec{x}_P = \frac{R_0}{\|\vec{x}\|}\vec{x}$$

be the nearest projection of the point $\vec{x}$ $(\vec{x} \neq 0)$ on the sphere $\|\vec{x}\| = R_0$. For the solution $\vec{x} = \vec{x}(t)$ when $\|\vec{x}(t)\| \geq 2R_0$ we have

$$\frac{1}{2}\frac{dv}{dt} = (A\vec{f}(\vec{x}, t), \vec{x}) =$$

$$= ([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{x}_P, t)], (\vec{x} - \vec{x}_P)) + ([A\vec{f}(\vec{x}_P, t) - A\vec{f}(\vec{0}, t)], (\vec{x} - \vec{x}_P)) +$$

$$+ ([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)], \vec{x}_P) + (A\vec{f}(\vec{0}, t), \vec{x}).$$

Hence, using the principal lemma, we have

$$([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{x}_P, t)], (\vec{x} - \vec{x}_P)) \leqslant \max_{\theta} \Lambda(\vec{x}_P + \theta(\vec{x} - \vec{x}_P), t)\|\vec{x} - \vec{x}_P\|^2 \leqslant$$

$$\leqslant -\alpha\|\vec{x} - \vec{x}_P\|^2 = -\alpha(\|\vec{x}\| - R_0)^2 = -\alpha\left(\frac{\|\vec{x}\|}{2} + \frac{\|\vec{x}\|}{2} - R_0\right)^2 \leqslant -\frac{\alpha}{4}\|\vec{x}\|^2,$$

where

$$0 \leqslant \theta \leqslant 1 \text{ and } \|\vec{x}\| \geqslant 2R_0.$$

Similarly, using inequality (6b), we have

$$([A\vec{f}(\vec{x}_P, t) - A\vec{f}(\vec{0}, t)], (\vec{x} - \vec{x}_P)) = ([A\vec{f}(\vec{x}_P, t) - A\vec{f}(\vec{0}, t)], \vec{x}_P)\left(\frac{\|\vec{x}\|}{R_0} - 1\right) \leqslant$$

$$\leqslant \beta\|\vec{x}_P\|^2\left(\frac{\|\vec{x}\|}{R_0} - 1\right) \leqslant \beta R_0\|\vec{x}\|.$$

Further, bearing in mind that

$$\Lambda(x, t) \leqslant \max(-\alpha, \beta) = \beta$$

follows from (6a) and (6b) when $\vec{x} \in E^n$ and $t \in I^+$, we find

$$([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t)], \vec{x}_P) = ([A\vec{f}(\vec{x}, t) - A\vec{f}(\vec{0}, t), \vec{x}) \frac{R_0}{\|\vec{x}\|} \leqslant$$

$$\leqslant \beta \|\vec{x}\|^2 \frac{R_0}{\|\vec{x}\|} = \beta R_0 \|\vec{x}\|.$$

Finally,

$$(A\vec{f}(\vec{0}, t), \vec{x}) \leqslant \|A\| \|\vec{f}(\vec{0}, t)\| \|\vec{x}\| \leqslant \rho c \|\vec{x}\|,$$

where $\rho = ||A||$. Therefore,

$$\frac{1}{2}\frac{dv}{dt} < -\frac{\alpha}{4}\|\vec{x}\|^2 + (2\beta R_0 + \rho c)\|\vec{x}\|,$$

or

$$\frac{dv}{dt} < -\frac{\alpha}{4}\|\vec{x}\|^2 - \left\{\frac{\alpha}{4}\|\vec{x}\| - 2(2\beta R_0 + \rho c)\right\}\|\vec{x}\| \leqslant -\frac{\alpha}{4}\|\vec{x}\|^2, \tag{14}$$

if only $||x|| \geq \max\left[2R_0, \frac{8(2\beta R_0 + \rho c)}{\alpha}\right] = R_1$.

Since

$$a\|\vec{x}\|^2 \leqslant v(\vec{x}) \leqslant b\|\vec{x}\|^2,$$

where $a = \lambda_{min}(A)$ and $b = \lambda_{max}(A)$, then from inequality (14) when $v \geq v_0 = bR_1^2$ we obtain

$$\frac{dv}{dt} \leqslant -\frac{\alpha}{4b} v.$$

Hence

$$v[\vec{x}(t)] \leqslant v[\vec{x}(t_0)] e^{-\frac{\alpha}{4b}(t - t_0)}, \tag{15}$$

if $v[\vec{x}(t)] \geq v_0$.

Therefore, when $t \geq T(\vec{x})$ for each solution $\vec{x}(t)$ the inequality $v[\vec{x}(t)] \leq v_0$ will be satisfied continuously, that is,

$$\|\vec{x}(t)\| \leqslant \sqrt{\frac{v[\vec{x}(t)]}{a}} \leqslant \sqrt{\frac{v_0}{a}} = \sqrt{\frac{b}{a}} R_1 = R$$

for $T(\vec{x}) \leq t < +\infty$.

<u>Theorem 2b.</u> If instead of inequalities (6a) and (6b) the inequality

$$\Lambda(\vec{x}, t) \leqslant -\varepsilon(r) < 0 \tag{6c}$$

is satisfied when $||\vec{x}|| \geq r > 0$ and $\underline{t} < t < +\infty$*, where $\alpha(r)$ is a monotone nondecreasing function of r ($\underline{t} < t < +\infty$) and, in addition,

$$\vec{f}(\vec{0}, t) \equiv \vec{0},$$

then the trivial solution $\vec{x} \equiv 0$ of System (5) is stable in the large when $t \to +\infty$.

<u>Proof</u>. Let

$$v(\vec{x}) = (A\vec{x}, \vec{x}).$$

Let $\varepsilon > 0$ arbitrarily and $||\vec{x}(t)|| \geq \varepsilon$. On the basis of the reasoning of theorem 2a, taking into account that we may take $\beta = 0$ and $c = 0$, we have

$$\frac{dv}{dt} \leqslant -\frac{1}{2}\alpha\left(\frac{\varepsilon}{2}\right)\|\vec{x}(t)\|^2 < 0$$

when $||\vec{x}(t)|| \geq \varepsilon$.

Thus $\vec{v}(\vec{x})$ is a positively defined function having a negatively defined derivative $\frac{dv}{dt}$ on the strength of System (5). Therefore, on the basis of Lyapunov's theorem, the solution $\vec{x} \equiv \vec{0}$ is asymptotically stable when $t \to +\infty$.

The asymptotic stability of the solution $\vec{x} \equiv \vec{0}$ holds for any initial perturbations, since from an inequality similar to (15) for each solution $\vec{x}(t)$ we obtain

$$\|\vec{x}(T)\| = \varepsilon \quad \text{at some } T > t_0,$$

and, therefore, $\lim\limits_{t \to +\infty} \vec{x}(t) = 0$.

<u>Remark</u>. Theorem 2b generalizes Krasovskiy's result [5].

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* When $r = 0$ we have $\Lambda(0,t) \leqslant 0$.

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